An Exploration of Nilpotent Group

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Abstract

We split our discussion to several parts. In the first part we will have a quick review of Sylow's Theorems, then we will present two proofs of the theorem that nilpotent group is a product of Sylow p-subgroups, the first relies on a bit complicated property on nilpotent group which we assume, while the second is built from one simple definition of the nilpotent group.

1 Warm Up: Review of Sylow

- Sylow I: $Syl_p(G) \neq \emptyset$
- Sylow II: P,Q are conjugate if both Sylow p-subgroup
  Key: Let Q acting on cosets of P in G by left multi. By Orbit-Stab thm, $|orb(gP)| = 1, p, ..., p^a$. On the other hand, there are $m = |G|/|P|$ (coprime to p) orbits. So there's a singleton orbit: $\{gP\}$
  Thus $\forall q \in Q, q(gP) = gP \Rightarrow g^{-1}qg \in P \Rightarrow g^{-1}Qg \subseteq P$. As same size, $P = Q$.
- Sylow III: $n_p \mid |G|$, $n_p \equiv 1 \pmod p$.
  Key: G acting on $Syl_p(G)$ by conjugation, one orbit, n_p divides G.
  P acts on $Syl_p(G)$ by conjugation, size of orbit $1, p, ..., p^a$. Clearly $\{P\}$ singleton orbit. If Q singleton, then $\forall p \in P, pQp^{-1} = Q$. So $P \subseteq N_G(Q)$.
  Since $N_G(Q), p^a \mid |N_G(Q)|, |N_G(Q)| \mid p^a m$, we know P,Q are Sylow p-subgroups of $N_G(Q)$ so that they are conjugate. Also, $Q \cong N_G(Q)$. So $P = Q$. One word, Q a Sylow p-subgroup is the only Sylow p-subgroup of its normalizer.

2 Nilpotent Group: First Visit

The frequent-seen definition of Nilpotent Group is a bit complicated, so we will not present at first. Nilpotent Group is important as, from Galois Theory, we can prove there's no formula for the quintic by considering groups like nilpotent groups. It is also very important in the classification of groups. One can think it slightly more complicated than abelian groups.
In this section We will assume a single key property of nilpotent groups:

- If G a nilpotent group, $H \leq G$ is a proper subgroup, then $H \leq N_G(H)$ is a strict containment. (†)

By (†), if $N_G(H) = H$, then $H = G$.
We are going to show that any finite nilpotent group is a product of p-groups.

Lemma 1. *Let G be a finite group and P a Sylow p-subgroup. Then $N_G(N_G(P)) = N_G(P)$.*

Proof. Clearly $N_G(P) \subseteq N_G(N_G(P))$.
$\forall g \in N_G(N_G(P)$, $gN_G(P)g^{-1} = N_G(P)$. As $P \leq N_G(P)$, we know $gPg^{-1} \leq N_G(P)$. Since P is a normal Sylow subgroup in $N_G(P)$, we must have $gPg^{-1} = P$. (c.f. to Sylow III proof) □

Corollary 1. *If G is a finite nilpotent group, then any Sylow subgroup is normal.*

Proof. $N_G(P)$ must be G by (†) and lemma 1. □

Thus we can show that:

Theorem 1. *If G is a finite nilpotent group, then it has single normal subgroup $P_1, P_2, ..., P_K$ and $G \cong P_1 \times ... \times P_k$.*

Proof. Write $|G| = p_1^{a_1}...p_k^{a_k}$, let $P_1, ..., P_k$ be Sylow subgroups with order $p_1, ..., p_k$. By lemma 1 and Corollary 1, all of them are normal in G. They are have trivial intersection, so $G \cong P_1 \times ... \times P_k$.
Note: A trick for induction step of the last line: $P_1P_2P_3 = (P_1P_2)P_3 \cong (P_1P_2) \times P_3$ requires $P_1P_2 \cong G$, which is trivial once we write it as $gP_1P_2g^{-1} = gP_1g^{-1}gP_2g^{-1}$. □

Our next step is to prove the theorem via an alternative method without using (†). We start from some properties that all groups share.

3 Frattini Argument, Commutator Subgroup

Frattini Argument
Let $K \cong G$ and P a Sylow p-subgroup of K. Then $G = N_G(P)K$.

Proof. We have P, $g^{-1}Pg = P$ are both p-subgroups in K, hence conjugate, i.e. k, $kPk^{-1} = gPg^{-1} \Rightarrow gk^{-1} \in N_G(P)$. □

It's not the result that's important, but the argument itself.

Commutator(Derived) Subgroup
For $g, h \in G$, define commutator as $[g, h] = ghg^{-1}h^{-1}$. Define G' (sometimes written as $[G, G]$ to be the subgroup generated by all $[g, h]$ for $g, h \in G$.

Lemma 2. *$G' \trianglelefteq G$. For $N \trianglelefteq G$, G/N is abelian if and only if $G' \subseteq N$. In particular, G/G' is abelian.*

Proof. The first statement is trivial if we notice that $g^{-1}x_1x_2...x_ng = (g^{-1}x_1g)(g^{-1}x_2g)...(g^{-1}x_ng)$.
($\Leftarrow$) If $G' \subseteq N$, $\forall a, b \in G$ We have $(ab)(ab)^{-1} \in G' \in N \Rightarrow abN = baN \Rightarrow (aN)(bN) = (bN)(aN)$
($\Rightarrow$) The direct inverse argument. □

We should also define:

Frattini Subgroup

$$\Phi = \bigcap_{M \leq G} M,$$

where M are all maximal subgroups of G.
Note: A maximal subgroup M of G is a subgroup that's not contained in any proper subgroup of G.

Lemma 3. *$\Phi \trianglelefteq G$ for any group G.*

Proof. Conjugation by any $g \in G$ induces an automorphism of G. Automorphisms must send a miximal subgroup to a maximal subgroup (not necessarily itself), so an automorphism shuffles maximal subgroups while the set of maximal subgroups are unchanged. So Φ is fixed. Thus Φ is normal in G. (It's actually characteristic by this argument). □

4 Nilpotent Group: Revisit

Now we use one simple definition of Nilpotent Group (this definition can be proved to be equivalent of other definitions):

Nilpotent Groups are the groups whose maximal subgroups are normal.

Note: by this we avoid using (†). One can easily see that (†) implies maximal subgroups of nilpotent groups are normal.

Lemma 4. *In a nilpotent group, $G' \subseteq M$, where M is any maximal subgroup, G' is its commutator subgroup. Hence $G' \subseteq \Phi$.*

Proof. Using subgroup correspondence, there are only two subgroups of G containing M, i.e. G and M. Thus G/M has only two subgroups, itself and identity.
By Cauchy's Theorem, G/M can only be a cyclic group f order p. So G/M is abelian, hence $G' \subseteq M$ by lemma 2. So $G' \subseteq \Phi$ by definition. □

Theorem 2. *Any Sylow p-subgroup is normal in G, where G is a finite Nilpotent Group. Thus, G is the product of Sylow subgroups.*

Proof. Let P be a Sylow p-subgroup of G. Let $K = \Phi P$. The product makes sense as $\Phi \trianglelefteq G$.
Let $g \in G$, $k \in K$. Then by lemma 4, $gkg^{-1}k^{-1} \in G' \in \Phi \subseteq K \Rightarrow gkg^{-1} \in K$. So $K \trianglelefteq G$.
P is a Sylow p-subgroup of K, as $P \leq K \leq G \Rightarrow p^a \mid |K|$ and $|K| \mid p^a m$.
Now We apply the Frattini Argument: $G = KN_G(P) = \Phi P N_G(P) = \Phi N_G(P)$. We shall show that $N_G(P) = G$.
If $N_G(P) \neq G$, then it must be contained in some maximal subgroup: if $N_G(P)$ is itself maximal, stop. If not, it is strictly contained in some subgroup $H_1 \leq G$, if H_1 maximal we stop, otherwise H_1 strictly contained in H_2... As G finite it will stop. So we can find a maximal subgroup H such that $N_G(P) \leq H$. By definition of Φ, $\Phi \subseteq H$. So $N_G(P)\Phi \subseteq H \neq G$, a contradiction.
Thus, $N_G(P) = G$. The result soon follows. □