Characteristic Group, Group of Order 60

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Definition 1. *A subgroup H of a group G is called a characteristic subgroup if for every automorphism ϕ of G, one has $\phi(H) = H$; then write H char G.*

Then we have:

Proposition 1.

1. *If H is characteristic in G, then H is a normal subgroup of G.*
2. *If H is the unique subgroup of G of a given order, then H is characteristic in G.*
3. *If K is characteristic in H and H is normal in G, then K is a normal subgroup in G*

Proof. 1 and 2 are trivial.
For 3, consider ϕ_g defined by $\phi_g(x) = g^{-1}xg$. Then $\phi_g(H) = H$. Then restriction $\phi_{g|H}$ is an automorphism of H.
Then we have $\phi_{g|H}(K) = K$ as K is characteristic in H. Since this works for all $g \in G$, we know K in normal in G. □

We now see a corollary of this.

Lemma 1. *A group G of order 30 has a normal subgroup of order 5.*

Proof. Suppose not. Then by Sylow, it must have 6 subgroup of order 5. Then $n_3 = 1$. Let P be this Sylow 3-subgroup. Then G/H has order 10, so G/H has a unique normal subgroup of order 5 by Cauchy. By normal subgroup correspondence, G has a normal subgroup of order 15. A group of order 15 must be cyclic, and contain a characteristic subgroup of order 5. Then the result follows. □

Actually, we notice that:

Lemma 2. *If G/N has a normal Sylow p-subgroup, and any group of order $p|N|$ is such that it must has a normal Sylow p-subgroup, then G has a normal p-subgroup.*

Proof. By normal subgroup correspondence, G has a normal group K with order $p|N|$. As any group of order $p|N|$ has a normal Sylow p-subgroup, by Proposition 1.3, we are done. □

Corollary 1. *Let G be a group of order 60 with more than one Sylow 5-subgorup. Then G is simple.*

Proof. By Sylow, if $n_5 \neq 1$, then $n_5 = 6$. If N is normal in G,
(i) 5 divides $|N|$. Then N has a Sylow 5-subgroup. Furthermore, since all Sylow 5-subgroups are conjugate, N must have all 6 Sylow 5-subgroups. As $|N|$ divides 60, we must have $|N| = 30$. By Lemma 1, we are done.
(ii) 5 does not divides $|N|$. Then $|N| = 2, 4, 6$ or 12.
If $|N| = 6$ (or 12), it has a unique normal Sylow 3(or 4)-subgroup, and hence it's characteristic in N. By Proposition 1.1.3, this characteristic subgroup is also normal in G. So we just need to consider 2,3 or 4.
For $|N| = 2$, $|G/N| = 30$, so G/N has a normal subgroup of order 5. By normal subgroup correspondence, there's a normal subgroup of order 10 in G. So by case (i) we are done. The case for $|N| = 3$ or 4 are the same.
Thus, either case G is simple. □

Honestly, we can further prove that G is isomorphic to A_5.

Theorem 1. *Let G be a group of order 60 with more than one Sylow 5-subgroup. Then $G \equiv A_5$.*

Proof. By Corollary 1, we know than G must be simple. By considering the conjugation action of G acting on the set of 6 Sylow 5-subgroups, we know that $G \leq A_6$ of index 6.
Then, by considering the left-multiplication action of A_6 on A_6/G, this induces a homomorphism from A_6 to $Sym(A_6/G)$. Obviously we can further restrict it to a homomorphism from A_6 to A_6. As G is fixed, we must have that this homomorphism take G to a subgroup of A_5. Since order G equals order of A_5, we have G is actually A_5. □